Back to Black

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Abstract

In this paper we seek to develop an active investment strategy that will allow us to outperform a relevant broad-market average. We seek outperformance not only on a net returns basis but also on a risk-adjusted basis. In order to do so we apply two theories which are usually at odds with each other – efficient market theory and behavioral finance. We find that while one is usually used to find faults in the other, if we combine them together we can develop a profitable, active investment system.

Recognizing that complexity in trading systems is often the cause of problems rather than the solution we strive to make the system as simple and accessible as possible. We constrain our analysis to the US equities market – it is the most popular market amongst active managers for its deep liquidity, breadth and data availability.

In order to build the system we first determine an adequate asset pricing framework, one that can at least guide us towards what an equilibrium price for a given security might represent. We then develop a method to gauge the deviations caused by over and under reaction and infer the distance between what the current price is and what the equilibrium price should be. If we find that the distance is significant enough we position ourselves to
take advantage of the situation. We test the system using a large amount of data gathered from various sources. Using almost 7.5 million observations we measure a wide range of performance metrics on several different dimensions.

We were successful. We have identified an active management approach that shows significant outperformance against its benchmark across numerous metrics. We then worked backwards and attempted to break our system by varying its parameters through a wide range of values to identify potential pitfalls in our analysis. The system was robust enough to survive. We evaluate its performance across a range of macro-scale environments and offer guidance to its proper deployment.

The system offers strong support to proponents of active management. It shows that bearing the right kind of idiosyncratic risk can yield results beyond what modern portfolio theory and the efficient market hypothesis would have you believe. As the selection of risk exposure is inherently an active pursuit, simply following a passive investment strategy would not be able to yield comparative results.

An appendix is also provided containing some mathematical results that add rigor to our work. It is NOT required reading and can be safely ignored.
**Introduction**

Efficient market theory states that prices always accurately reflect the fundamentals. On the other hand, behavioral economics says that prices reflect individual biases which leads to over and under-reactions to the fundamentals. Put together, the result is that prices oscillate back and forth but eventually converge to an equilibrium dictated at least in part by the fundamentals. At any one time there are various degrees of efficiency in the market – a varying number of stocks will be over or underpriced to different extents. There is thus an opportunity to generate profits by buying the underpriced issues and selling the overpriced ones. It is on this premise that we will attempt to build a profitable trading system.

We begin by presenting an asset pricing framework, one which will allow us to identify which stocks might be over or under priced. We then analyze whether or not a trading system can be profitably implemented using this information. We seek to determine what characteristics such a system might present and whether there are corners of the market where it is especially profitable. Pulling all this information together we develop a set of simple yet comprehensive trading rules which a practitioner could apply
to extract excess return from the US equities market. We then validate and stress-test the system.

**Determining Asset Pricing**

It is natural to begin by asking how to determine whether or not a stock is fairly priced or whether any deviation is present. To answer that question we will decompose the returns into systematic and idiosyncratic risk factors and then model the idiosyncratic part using an appropriate algorithm. To separate the two we first need to determine an appropriate set of systematic risk factors. Several methods are available, however we will defer to the tried-and-true Capital Asset Pricing Model (CAPM). While not the most sophisticated or the most theoretically rigorous method, it is definitely one of the simplest and most well-known. For our purposes it is a good enough proxy for systematic risk and in the spirit of parsimony and to avoid unnecessary complexity, we will use it.

Our initial model thus becomes:

$$ R_{asset} = \alpha + \beta (R_{mkt} - R_{rf}) + \epsilon $$  \hspace{1cm} (1)

The return for any asset can be expressed as an additive function of its relationship to market returns net of the risk-free rate plus an idiosyncratic
component, $\alpha + \varepsilon$. In this paper we will focus on the US equities market. The access to data is excellent, it has deep liquidity and wide breadth – for those reasons it is a very popular market for systematic traders. As such, we will use the Russell 3000 index with dividends reinvested as a proxy for the market. This index covers 98% of market cap of the United States.

The idiosyncratic component is made up of two parts, $\alpha + \varepsilon$. $\alpha$ represents the excess return above the exposure to the market. It has been studied extensively in the past and we will not focus on it here. Going forward we will assume it is zero and concentrate exclusively on the residual process $\varepsilon$. It is usually assumed to be a zero-mean, normally distributed random variable however we will break with tradition and instead model it as a mean-reverting stochastic process.

$$\varepsilon_t := dX_t = \lambda(\mu - X_t)dt + \sigma dB_t$$ (2)

The technical term for this kind of equation is an Ornstein-Uhlenbeck (OU) process. It has a rich history in physics in areas such as fluid dynamics and dampened spring mechanics. The long term mean of this process is $\mu$ - it always tends to revert to this value. Figure 1 illustrates this behavior. The process is started at different values and then quickly reverts to the mean.
The variable $\lambda$ represents the speed of mean-reversion. The expected time to revert back to the mean is $1 / \lambda$. Variability in the process is expressed as the parameter $\sigma$ applied to a white noise Brownian motion process $dB_t$.

![Figure 1: Several paths of a discretized OU process with different initial values. Parameters: $\lambda = 0.1$, $\mu = 20$, $\sigma = 1$, $\delta = 0.25$](image)

The parameters are unique to each stock in the universe.

This model produces simple dynamics. A stock will exhibit returns proportional to its exposure to the market plus an idiosyncratic component. If the resulting price is too far away from equilibrium it will revert back. Figure 2 illustrates an idealized situation. By estimating the parameters of
the residual process we will be able to identify the peaks and troughs and position ourselves to benefit from the resultant mean reversion.

**Figure 2: Idealized model dynamics**

**Parameter Estimation**

In order to construct valid trading signals we first need to estimate the parameters to the OU process driving the residual returns. Since the OU process is defined in continuous time we must first compute a suitable discretization. Several generic methods are available (Euler-Maruyama, Runge-Kutta, etc.) however building on the work of Gillespie (1996) we will use the following iterative equation:
\[ X_{t+1} = X_t e^{-\lambda \delta} + \mu (1 - e^{-\lambda \delta}) + \sigma \sqrt{\frac{1-e^{-2\lambda \delta}}{2\lambda}} N(0,1) \]  

(3)

where \( \delta \) is the time-step and the other variables are defined as before.

Recognizing that this is really an auto-regressive model with lag one (AR(1) model) we can estimate the parameters through an ordinary least squares regression or maximum likelihood. We will use the regression method with an estimation window of 60 trading days, which corresponds to approximately one fiscal quarter.

For any given time series of stock returns, the method works as follows:

1. Regress the stock returns on the returns of the market less the risk-free rate
2. Create the discrete OU process by cumulatively summing the residuals:
   \[ X_t = \sum_{i=1}^{t} \epsilon_t, \quad t = 1 \ldots 60 \]  

(4)

3. Run the one-lag regression model on the results from step 2:
   \[ X_{t+1} = a + bX_t + \zeta_{t+1}, \quad t = 1 \ldots 59 \]  

(5)
4. The parameters are given by:

\[ \lambda = -\frac{\ln(b)}{\delta} \]  

(6)

\[ \mu = \frac{a}{1-b} \]  

(7)

\[ \sigma = \frac{\sqrt{\text{Variance}(\zeta) \cdot 2\lambda}}{1-b^2} \]  

(8)

\[ \sigma_{eq} = \frac{\sqrt{\text{Variance}(\zeta)}}{1-b^2} \]  

(9)

One of the interesting properties of this model is that although it is mathematically very rigorous, once developed, all the relevant quantities can be trivially computed using a simple spreadsheet application.

**Signaling**

From the estimated parameters we can infer how far away from the mean is the current value of the process. We call this distance MS (short for Model Score) and express it in units of standard deviation:
Because of the regression computation $X_{60} = 0$ and so the MS becomes:

$$MS = \frac{X_t - \mu}{\sigma_{eq}} \quad (10)$$

$$MS = -\frac{\mu}{\sigma_{eq}} \quad (11)$$

A high positive value indicates the stock is overbought and vice-versa.

**Model Analysis**

Now that we have developed a proper model, we ask ourselves what every practitioner does – does this actually work? Can we make money with this?

In order to assess the efficacy of the model we will first need data. For this purpose we have collected daily data snapshots\(^1\) on over 5,800 US stocks as of January 1\(^{st}\), 2005. In total we have 7,648,150 stock-days. We split this data into two sets – the first set runs from January 1\(^{st}\), 2005 until January 1\(^{st}\), 2010 and the second set covers the rest of the time until January 1\(^{st}\), 2014. The data is split to allow us to first assess and develop the system and to then look out-of-sample for validation. We want our analysis to be as broad as possible however we do want to be mindful of

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\(^1\) The data is taken from the Ford database, joined with Thomson Reuters and Capital IQ. It is survivorship bias free and includes all the firms having traded on the major US exchanges since January 1\(^{st}\), 2005.
the liquidity such a strategy requires. For this reason we exclude stocks with a market cap of less than 100M from the analysis. Further we will impose additional constraints on liquidity and study their impact on our findings.

For the first part, we will conduct back-tests using the portfolio approach. For every day in the first data set, we divide the universe in deciles according to their MS. We then form equally-weighted portfolios comprising of all the stocks in each decile and hold them for several days, calculating forward returns. For instance, if the holding period is one day, if the MS was calculated on a Monday, we buy the stock on Tuesday, hold it until Wednesday’s close and compute the return.

The results, with comment, are presented in the following figures.

The system performs admirably. As we expected, stocks deemed underpriced (low deciles) outperform while stocks deemed overpriced (high deciles) underperform. Note that we report returns net of the benchmark (Russell 3000) – any system worth pursuing must be able to at least beat a passive ETF.
After five days the most underpriced stocks still show massive outperformance however less so than after one day. This is not surprising – as we move through time we expect that stocks will revert to their equilibrium price. From the equation defining our model we also know that the magnitude of the returns is dependent on the distance from equilibrium.

Taken together, these two facts indicate that we should indeed expect a slight drop in daily performance numbers for any given name as we move forward through time and the stock reverts. This implies that when we identify an opportunity to enter a trade, we should be quick and decisive.
Figure 4: Back-test results, 5-day holding period, Jan 1st 2005 – Jan 1st, 2010

Notice that the change in the magnitude of the returns in the tails of the distribution is roughly symmetric on both sides. The return of decile one is less by a very similar amount that the return of decile ten is greater. The same applies to deciles two and nine and three and eight. In the middle there is very little change. This is consistent with a number of previous papers that stipulate that slight deviations from equilibrium can persist as transaction costs render the trades required to bring them in line not profitable.
Figure 5 tells a very similar story. The strategy continues to generate above-market results for up to (at least) two weeks after the signal date. The table below provides some additional characteristics of the decile portfolios. Since the selection criteria were identical in all three graphs, it applies to all of them.
Table 1: Median values of back-test portfolios

<table>
<thead>
<tr>
<th>Model Score (MS)</th>
<th>-1.67</th>
<th>-1.11</th>
<th>-0.77</th>
<th>-0.46</th>
<th>-0.16</th>
<th>0.13</th>
<th>0.43</th>
<th>0.72</th>
<th>1.06</th>
<th>1.61</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constituents</td>
<td>354</td>
<td>354</td>
<td>354</td>
<td>354</td>
<td>354</td>
<td>354</td>
<td>354</td>
<td>354</td>
<td>354</td>
<td>354</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>42.5%</td>
<td>43.1%</td>
<td>42.2%</td>
<td>41.7%</td>
<td>41.6%</td>
<td>41.6%</td>
<td>42.3%</td>
<td>42.9%</td>
<td>43.2%</td>
<td>42.8%</td>
</tr>
<tr>
<td>Beta</td>
<td>0.82</td>
<td>0.81</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
</tr>
<tr>
<td>Market Cap (M$)</td>
<td>3,751</td>
<td>4,024</td>
<td>4,074</td>
<td>4,096</td>
<td>4,072</td>
<td>4,064</td>
<td>4,072</td>
<td>4,107</td>
<td>4,127</td>
<td>3,876</td>
</tr>
</tbody>
</table>

The data table reveals some interesting details. The system shows no bias towards volatility, beta or market cap. The median values for each of these variables show very little variance between the different deciles. This is a great property to have as it doesn’t restrict us to operate in a particular corner of the market.

Next we work our way down the fractal chain and divide the universe into finer slices. Doing so provides a measure of robustness to our system. As we increase the granularity the leftmost and rightmost quantiles contain more extreme values, which we expect to show returns of progressively greater magnitude.
Figure 6: Back-test results, increased sampling granularity, Jan 1\textsuperscript{st} 2005 – Jan 1\textsuperscript{st}, 2010

Figure 6 validates our expectations. More extreme values do indeed lead to higher returns.

**Back to Black - System Implementation**

With the information we’ve collected so far we can now proceed to the next step, which is the development of a complete set of rules and heuristics that combined together form a profitable, systematic trading strategy that
can be readily applied by practitioners. Given what we now know, we can outline a series of steps that will allow us to reap the benefits of the information we have constructed.

First, an initial portfolio is first set up by buying the appropriate amount of the most underpriced stocks. The holdings are then analyzed daily according to the following rules:

**I. After Market Close**

1. Determine the investment universe. This is all stocks with a market cap of at least 100M
2. Using the latest available data, calculate the MS for every stock as described above
3. Flag to sell any stocks currently held whose current value exceeds the exit threshold
4. For every stock flagged to sell, flag to buy one stock starting from the most underpriced

**II. At Next Market Open**

1. Execute trades
2. Hold portfolio until market close
The previous list of steps contains two parameters that we must configure: how many stocks to hold and the sell threshold. We want to always maintain a good level of diversification but still maintain a manageable portfolio. As such we will hold a portfolio of 50 stocks. We will use an exit threshold of -0.5. We select this number in deference to our earlier analysis – we know the stocks will revert but not completely, only to a level where it would make sense for new buyers to come in and close the pricing gap. When the MS of a stock in the portfolio reaches that level, it will be sold at the next possible opportunity. This is not an optimized parameter set, merely one that is intuitively plausible. We will have more to say about the parameter choices in the next section.

Putting it all together we perform a simulated run of our strategy. The rules are followed every day and the value of the resulting portfolio is recorded daily. Figure 6 displays the value of $1 invested in the system and a benchmark portfolio, in this case the Russell 3000 Total Return index.
Confirming our previous analysis, the system strongly outperforms the market. The total value of our investment is $2.26, a total return of 126% after 4 years, or 22.6% yearly. In comparison, the benchmark portfolio is only worth $1.79, a yearly return of 15.7%.

Table 2 contains a detailed breakdown of several key metrics by year. The first three years the system easily beats the benchmark however the last year has been difficult. Note however that while relative performance was negative, the gross return was still almost 23%.
<table>
<thead>
<tr>
<th></th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
<th>2013</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Returns</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Portfolio Return</td>
<td>28.0%</td>
<td>13.4%</td>
<td>26.7%</td>
<td>22.9%</td>
</tr>
<tr>
<td>Benchmark Return</td>
<td>14.7%</td>
<td>1.0%</td>
<td>16.4%</td>
<td>33.0%</td>
</tr>
<tr>
<td>Net Return</td>
<td>13.3%</td>
<td>12.4%</td>
<td>10.3%</td>
<td>-10.1%</td>
</tr>
<tr>
<td><strong>Performance Measures</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beta</td>
<td>1.17</td>
<td>1.24</td>
<td>1.21</td>
<td>1.17</td>
</tr>
<tr>
<td>Correlation</td>
<td>0.93</td>
<td>0.95</td>
<td>0.92</td>
<td>0.89</td>
</tr>
<tr>
<td>Portfolio Sharpe Ratio</td>
<td>1.19</td>
<td>0.42</td>
<td>1.54</td>
<td>1.53</td>
</tr>
<tr>
<td>Benchmark Sharpe Ratio</td>
<td>0.78</td>
<td>0.04</td>
<td>1.25</td>
<td>2.90</td>
</tr>
<tr>
<td>Treynor Ratio</td>
<td>0.24</td>
<td>0.11</td>
<td>0.22</td>
<td>0.20</td>
</tr>
<tr>
<td>M Squared</td>
<td>0.35</td>
<td>0.18</td>
<td>0.36</td>
<td>0.30</td>
</tr>
<tr>
<td>Information Ratio</td>
<td>1.04</td>
<td>1.07</td>
<td>0.77</td>
<td>-1.72</td>
</tr>
<tr>
<td>Sortino Ratio</td>
<td>0.05</td>
<td>0.04</td>
<td>0.05</td>
<td>-0.04</td>
</tr>
<tr>
<td><strong>Risk Measures</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Portfolio Volatility</td>
<td>23.5%</td>
<td>32.0%</td>
<td>17.4%</td>
<td>15.0%</td>
</tr>
<tr>
<td>Benchmark Volatility</td>
<td>18.7%</td>
<td>24.4%</td>
<td>13.1%</td>
<td>11.4%</td>
</tr>
<tr>
<td>Tracking Error</td>
<td>9.1%</td>
<td>11.8%</td>
<td>7.5%</td>
<td>7.2%</td>
</tr>
<tr>
<td>Portfolio Max Drawdown</td>
<td>-18.9%</td>
<td>-25.7%</td>
<td>-14.6%</td>
<td>-6.4%</td>
</tr>
<tr>
<td><strong>Drawdown Recovery (Days)</strong></td>
<td>86</td>
<td>46</td>
<td>69</td>
<td>16</td>
</tr>
<tr>
<td>Benchmark Max Drawdown</td>
<td>-16.2%</td>
<td>-20.4%</td>
<td>-10.0%</td>
<td>-5.7%</td>
</tr>
<tr>
<td><strong>Drawdown Recovery (Days)</strong></td>
<td>87</td>
<td>63</td>
<td>53</td>
<td>12</td>
</tr>
<tr>
<td>Portfolio Pain Index</td>
<td>5.6%</td>
<td>5.2%</td>
<td>3.8%</td>
<td>1.7%</td>
</tr>
<tr>
<td>Benchmark Pain index</td>
<td>5.0%</td>
<td>5.8%</td>
<td>2.5%</td>
<td>1.0%</td>
</tr>
<tr>
<td><strong>Up / Down Ratios</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Batting Average</td>
<td>53.6%</td>
<td>51.6%</td>
<td>47.6%</td>
<td>48.4%</td>
</tr>
<tr>
<td>Up Market Capture</td>
<td>125.0%</td>
<td>133.5%</td>
<td>127.9%</td>
<td>106.6%</td>
</tr>
<tr>
<td>Down Market Capture</td>
<td>116.6%</td>
<td>124.5%</td>
<td>121.2%</td>
<td>123.6%</td>
</tr>
<tr>
<td>Up Market Ratio</td>
<td>0.89</td>
<td>0.91</td>
<td>0.85</td>
<td>0.83</td>
</tr>
<tr>
<td>Down Market Ratio</td>
<td>0.84</td>
<td>0.93</td>
<td>0.88</td>
<td>0.81</td>
</tr>
<tr>
<td>Up Market Percent</td>
<td>63.8%</td>
<td>69.6%</td>
<td>60.3%</td>
<td>51.4%</td>
</tr>
<tr>
<td>Down Market Percent</td>
<td>40.4%</td>
<td>29.8%</td>
<td>33.6%</td>
<td>44.2%</td>
</tr>
</tbody>
</table>

**Table 2: Detailed performance analysis**

In general, the system is slightly more risky than the market, running a beta of approximately 1.20 while tracking error is kept low.
Drawdowns are also higher than the market but time to recovery is, on average, pretty even. Despite the higher risk profile however, on a risk-adjusted basis the system performs very well. The Sharpe ratio is on average much higher than the market. The information ratio indicates that for the first two years of the test a manager utilizing this system would be in the top decile of active portfolio managers while in the third year the manager is still well into the top quartile. Even though there is some underperformance in the last year, the combined result is nevertheless highly positive.

**Parameter Variation and Stress Testing**

The system works with our initial parameter choices however this is not enough – it must be robust to a range of plausible values. We first examine the impact of modifying the sell threshold. As figure 7 shows, the system performs very well using a wide range of values.

Second we measure the impact of portfolio size on the total return of the system. Figure 8 displays the results. As with the sell threshold parameter, the system is robust against changes in portfolio size. It can be run with 20 stocks all the way through 60 without much incidence on performance.
As we previously mentioned, we want to keep an eye on liquidity and execution quality. We might not be able to get our full position at the same price, the market might have moved against us, the bid-ask spreads might have unexpectedly widened, etc. Microstructure effects are a reality in the markets and we must be able to cope with them. In order to gauge the impact of these kinds of perturbations we introduce friction costs in our
Figure 9: Impact of varying portfolio size on system performance

testing. On every trade that we make we adjust the price by a certain amount. This amount is expressed as a percentage of the share price in one direction. It is charged on every buy and every sell so the cost to rotate a position is double the indicated amount. The range of values we have tested reflect the friction one might experience in the real world trading on electronic marketplaces with a modicum of trading skill.

Figure 9 depicts the results. Friction costs (quoted in basis points, abbreviated bps) can have an important impact on the performance of the
system. It is important that care is taken to minimize the overall cost of trading however even with very high friction (20 bps) the system is still able to outperform the market.

![Chart: Growth of $1 Invested](image)

**Figure 10: Impact of friction costs on system performance**

**Market Regime Performance**

As every practitioner knows, the market is not homogenous – it goes through phases and cycles that can have very broad ramifications on the performance of certain investment styles. Having thoroughly tested the parts of our system that we can manipulate, we now take a step back and
take a macro view of the situation. We want to assess *when*, not just *how* to use the system. A properly timed market call can spell the difference between top and bottom quartile performance and we want to make sure we deploy our system at the right moment. Being out of the market at the right time can be as, if not more, profitable than being in at the right time.

We will use three indicators that can characterize different phases in the market. First is the VIX indicator, also known as the fear gauge of the market. The VIX can alert us to short-term investor sentiment. The second indicator is market exhaustion, which is measured by momentum. Very strong market performance is often followed by wide-scale retrenchment and vice-versa. Finally, we look at the now infamous risk-on / risk-off cycle. In the last several years this has been a popular topic and a number of methods were devised to measure which risk regime is currently dominating the market. There are two popular approaches – the first is by applying a hidden Markov model to a transformation of the market returns (for example the squared Mahalanobis distance) and the second is by looking at moving averages. We will use the second method, for a very simple reason – for all their complexity, theoretical richness and rigor, hidden Markov models have not been able to outperform the moving average approach in evaluating risk conditions in the equities market. The
The performance metric we use is the five day forward compounded return. This measure was chosen for two reasons: first it smooths out daily fluctuations while still taking into account short-term performance. Second, calculating forward returns rather than trailing allows us to act on our findings rather than simply explain a phenomenon.

**Reaction to Fear**

The methodology is simple. For every day in our sample set we calculate the returns for the next five days. We then sort them by the last available VIX observation before compounding began. If the VIX levels have any incidence on strategy performance then by plotting the sorted returns we will observe a pattern. Indeed, as figure 11 illustrates, there is a pattern - not in the returns but in the volatility! As the VIX climbs higher the strategy gets more volatile but a clear trend in performance isn’t visible. The takeaway is that if an active manager is interested in reducing the volatility of his fund during bouts of uncertainty, he would be well advised to shelf this system until a better entry point is available.
Market Exhaustion

We repeat the process except that this time instead of using VIX levels we use the market returns over the previous 30 days as our sorting metric.

In this case there is a clear trend in performance. There is a very high concentration of negative return bars on the left-hand side of figure 12, i.e. when the market has seen a strong decline in the previous month. At the same time there are much fewer negative bars on the right-hand side of the figure than we would expect if no relationship was present. This implies that
the system performs very well in bull markets but we should be careful when deploying it during a pronounced downturn.

![System Performance vs. Previous Market Performance](image)

**Figure 12:** System performance vs. previous market performance

**Market Risk Preference**

Applying our methodology one more time we plot forward returns vs. the ratio of the 50-day moving average of the market to the 200-day moving average. When the 50-day average is above the 200-day average we can say that the market is in a risk-on environment and vice versa. Once again
a clear trend emerges. The system tends to perform better in a risk-off environment. This is not surprising – during a risk-off phase fear is typically the prevailing emotion and overreactions are more likely to occur and to be bigger in magnitude. As our system is specifically designed to take advantage of these overreactions, this performance characteristic is exactly what we expect to see. Note however that the system doesn’t underperform in a risk-on environment – there is no discernable relationship there. Therefore we do not want to turn the system off during such times but instead we would increase the allocation to this strategy during a risk-off phase.

**System Performance vs. Market Risk Preference**

![Graph showing system performance vs. market risk preference](image-url)

*Figure 13: System performance during market risk preference environment*
Conclusion

Using a quantitative pricing framework we were able to infer an equilibrium expected value of a stock’s price. We then developed a model which allowed us to measure the deviation from this equilibrium. By combining this model with a simple set of rules we developed an active investment system that can outperform the market on a net return but also on a risk-adjusted basis. To build confidence in our results we varied the parameters through a wide range of values. The system maintains strong performance even in the face of such variation. This indicates robustness of design and increases our confidence towards its construction. We also analyzed performance during various macro-scale environments and offered advice on the proper deployment of the system.

This paper suggests that bearing the right kind of idiosyncratic risk can be rewarding – something which is precluded by modern portfolio theory. As the selection of idiosyncratic risk is by definition an active investment strategy, this paper also serves as strong support for the benefits of active investment management.
Appendix I – Analytic Solution of the Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck (OU) process is a type of mean-regressive Langevin equation widely used in physics and mathematical finance. In physics it can be used as a model of a relaxation process whereas in mathematical finance it is used to model the basis on commodity futures, interest rates and other situations where mean-reversion is desirable. It can be defined as the following linear (in the narrow sense), non-homogenous stochastic differential equation:

\[ dX_t = \lambda(\mu - X_t)dt + \sigma dB_t \]  \hspace{1cm} (A1)

where \( \lambda \) represents the speed of mean-reversion, \( \mu \) is the long-term mean and \( \sigma \) is the volatility. The coefficients are constant which ensures the existence and uniqueness of a solution.

In order to determine an analytic solution we must first solve the associated homogenous stochastic differential equation:

\[ dX_t = -\lambda X_t dt \]  \hspace{1cm} (A2)
To solve we make use of Itô’s lemma (alternatively called the Itô-Doeblin theorem):

\[ dg(t, x) = \frac{\partial g}{\partial t}(t, x)dt + \frac{\partial g}{\partial x}(t, x)dx + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, x)(dx)^2 \]  \hspace{1cm} (A3)

where \( g(t, x) \) is any twice-differentiable function continuous on \([0, \infty] \times \mathbb{R}\) and \((dx)^2\) is evaluated according to:

\[ dt \cdot dt = dt \cdot dB_t = 0, \hspace{0.2cm} (dB_t)^2 = dB_t = dt \]  \hspace{1cm} (A4)

Using \( \ln(X_t) \) as the \( g \)-function we obtain:

\[ d(\ln(X_t)) = \frac{dX_t}{X_t} - \frac{(dX_t)^2}{2X_t^2} \]  \hspace{1cm} (A5)

Making the appropriate substitutions and integrating,

\[ d(\ln(X_t)) = \frac{-\lambda X_t dt}{X_t} - \frac{\lambda^2 X_t^2 (dt)^2}{2X_t^2} = -\lambda dt \]  \hspace{1cm} (A6)

\[ \ln(X_t) - \ln(X_0) = -\int_0^t \lambda ds \Rightarrow X_t = X_0 e^{-\lambda t} \]  \hspace{1cm} (A7)
We set $X_0 = 1$ and denote the result as $\phi$. Applying the variation of parameters technique from ordinary differential equations we get:

$$X_t = Y_t \phi \Rightarrow Y_t = X_t \phi^{-1}, \quad \phi^{-1} = e^{\lambda t} \quad (A8)$$

Differentiating and applying the product rule of stochastic calculus,

$$dY_t = dX_t \phi^{-1} + X_t d\phi^{-1} \quad (A9)$$

where (once again applying Itô’s lemma),

$$d\phi^{-1} = \lambda e^{\lambda t} dt = \lambda \phi^{-1} dt \quad (A10)$$

which leads to:

$$dY_t = (\lambda(\mu - X_t)dt + \sigma dB_t)\phi^{-1} + \lambda X_t \phi^{-1} dt \quad (A11)$$

$$dY_t = \lambda \mu \phi^{-1} dt - \lambda X_t \phi^{-1} dt + \sigma \phi^{-1} dB_t + \lambda X_t \phi^{-1} dt$$

$$dY_t = \lambda \mu \phi^{-1} dt + \sigma \phi^{-1} dB_t$$

$$Y_t = Y_0 + \int_0^t \lambda \mu \phi^{-1} ds + \int_0^t \sigma \phi^{-1} dB_s \quad (A12)$$

Finally, setting $Y_0 = X_0$ and expanding,
The analytic solution to the Ornstein-Uhlenbeck process is therefore:

\[
X_t = \left( X_0 + \int_0^t \lambda \mu e^{-\lambda t} ds + \int_0^t \sigma e^{-\lambda t} dB_s \right) \phi
\]  
(A13)

\[
X_t = X_0 e^{-\lambda t} + \lambda \mu \int_0^t e^{\lambda(s-t)} ds + \sigma \int_0^t e^{\lambda(s-t)} dB_s
\]

\[
X_t = X_0 e^{-\lambda t} + \lambda \mu \left( \frac{1 - e^{-\lambda t}}{\lambda} \right) + \sigma \int_0^t e^{\lambda(s-t)} dB_s
\]  
(A14)

The analytic solution to the Ornstein-Uhlenbeck process is therefore:

\[
X_t = X_0 e^{-\lambda t} + \mu (1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda(s-t)} dB_s
\]  
(A15)

The process follows a normal distribution. The mean is:

\[
E[X_t] = E[X_0 e^{-\lambda t}] + E[\mu (1 - e^{-\lambda t})] + E[\sigma \int_0^t e^{\lambda(s-t)} dB_s]
\]  
(A16)

Using the martingale property of Itô integrals we can eliminate the stochastic term of the equation, yielding:

\[
E[X_t] = X_0 e^{-\lambda t} + \mu (1 - e^{-\lambda t})
\]  
(A17)
The variance is then:

\[
E[(X_t - E[X_t])^2]
\]

\[
= E[(X_0 e^{-\lambda t} + \mu(1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda(s-t)} dB_s] - X_0 e^{-\lambda t} - \mu(1 - e^{-\lambda t})]^2]
\]

\[
E[(X_t - E[X_t])^2] = E[(\int_0^t e^{\lambda(s-t)} dB_s)^2]
\]  (A19)

Using Itô’s isometry,

\[
E[(\int_0^t e^{-\lambda(s-t)} dB_s)^2] = E[(\sigma \int_0^t e^{2\lambda(s-t)} ds]^2]
\]  (A20)

\[
E[\sigma^2 \int_0^t e^{2\lambda(s-t)} ds] = \sigma^2 \int_0^t e^{2\lambda(s-t)} ds
\]

\[
\sigma^2 \int_0^t e^{2\lambda(s-t)} ds = \sigma^2 e^{-2\lambda t} \cdot \frac{e^{2\lambda s}}{2\lambda} \bigg|_0^t
\]  (A21)

which defines the variance as:
Taking the limit as $t \to \infty$ we get the equilibrium mean and variance, respectively:

$$
\lim_{t \to \infty} \left( X_0 e^{-\lambda t} + \mu (1 - e^{-\lambda t}) \right) = \mu \quad (A23)
$$

$$
\lim_{t \to \infty} \left( \sigma^2 \left( \frac{1 - e^{-2\lambda t}}{2\lambda} \right) \right) = \frac{\sigma^2}{2\lambda} \quad (A24)
$$
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“One of the funny things about the stock market is that every time one person buys, another sells, and both think they are astute.”

- William Feather